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PROBLEMS OF POTENTIAL THEORY

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1. *The Equations of Laplace and Poisson.*—As is well known, Bôcher considered the integral form of Laplace's equation:[1]

$$\int \frac{\partial u}{\partial n}\, ds = 0, \tag{1}$$

and showed that it was entirely equivalent to the differential form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

for functions u continuous with their *first* partial derivatives over any "Weierstrassian" region (as Borel would call it[2]); in fact he showed that any such solution of (1) possessed continuous derivatives of all orders and satisfied (2) at every point. One can go still further, and consider solutions of (1) which have merely summable derivatives, and of the first order, with practically the same result.

THEOREM I.—If the function u is what we shall call a "potential function for its gradient vector ∇u,"[3] the components of the latter being summable superficially in the Lebesgue sense, and if the equation

$$\int \nabla_n u\, ds = 0 \tag{1}$$

is satisfied for every curve of a certain class,[4] then the function u has merely unnecessary discontinuities, and when these are removed by changing the value of u in the points at most of a point set of superficial measure zero, the resulting function has continuous derivatives of all orders and satisfies (2) at every point.

Let us pass on to the equation

$$\int \frac{\partial u}{\partial n}\, ds = \int \rho\, d\sigma \tag{3}$$

in which a curvilinear integral on the left is equal to a superficial integral on the right, and this equality is a generalization of Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x,y). \tag{4}$$

But we need not limit ourselves in the right hand member to quantities which are the integrals of functions ρ; instead we may take the right hand member to be any sort of additive function of point sets $f(e)$[5] (not necessarily absolutely continuous as in (3) or even continuous), or any sort of additive function of curves in the plane $f(s)$,[6] of limited variation. More precisely, we consider the equation

$$\int \nabla_n u\, ds = F(s) \tag{3'}$$

in which $F(s)$ is an additive function of curves in the plane, of limited variation, with discontinuities "of the first kind."

The difference of any two solutions of (3′) is a solution of (1′), and, therefore, part of the study of (3′) is the study of a particular solution. Such a particular solution is found to be the function

$$u(M_1) = \frac{1}{2\pi}\int_{\Sigma} \log \frac{1}{r}\, df(e) \tag{5}$$

defined in terms of a Lebesgue-Stieltjes integral. In this equation M_1 is the point of coördinates x_1, y_1, M the point of coördinates x, y which is the argument of the integration, and r the distance between them; in measuring angles the sense of r is taken as M_1M. This function may be shown to be the potential function of a gradient vector of which the component in the direction α is given by

$$\nabla_\alpha u(M_1) = \frac{1}{2\pi}\int_{\Sigma} \frac{\cos \alpha r}{r}\, df(e). \tag{6}$$

The vector given by (6) may be shown to be a solution of (3′) for every curve of the class Γ (specified below).

In two dimensions, a Stieltjes integral may be defined equally well in terms of both sorts of functional, and a function of curves used as well as a function of point sets for the function of limited variation. And thus if Σ' is a region enclosed in Σ, with boundary S' (a curve of class Γ), we may speak of the function

$$u'(M_1) = \frac{1}{2\pi}\int_{\Sigma'} \log \frac{1}{r}\, dF(s), \tag{5'}$$

which is identical with the function given by (5) for all points interior to Σ', provided that the equality

$$F(s) = f(e) \ (e = \text{set of points inside } s), \tag{5''}$$

holds for any s of Γ on which $F(s)$ is continuous as a function of curves.

Methods of making correspondences of type (5″) will be discussed below.

2. *Various Forms of Green's Theorem.*—Let $F(s)$ and $G(s)$ be two functions of curves with discontinuities of the first kind, and $U(M)$ and $V(M)$ two solutions of the equations

$$\int \nabla_n U ds = F(s)$$

$$\int \nabla_n V ds = G(s),$$

which may be written in the form given by (5) or (5′) plus harmonic functions (i. e., solutions of (1′) after the unnecessary discontinuities have been removed). With respect to $F(s)$ and $G(s)$ let us introduce the symbols $T_F(s)$ and $T_G(s)$ to stand for their total variation functions. Then we have the following rather general result.

THEOREM II.—If the discontinuities of $F(s)$ and $G(s)$ are of the first kind, a sufficient condition that the equation

$$\int_\sigma (\nabla_x U \nabla_x V + \nabla_y U \nabla_y V) d\sigma = \int_\sigma U dG(s) - \int_s U \nabla_n V ds$$
$$= \int_\sigma V dF(s) - \int_s V \nabla_n U ds \qquad (7)$$

shall hold for all regions σ and all boundaries s of Γ, internal to Σ', is the existence of the quantity

$$\int_{\Sigma'} \log \frac{1}{r'_{M_1M}} dT_F(s) dT_G(s).$$

A vector φ which satisfies the equation corresponding to (3′)

$$\int_s \varphi_n ds = F(s) \qquad (8)$$

for every curve of Γ may be called a *polarization vector* for the distribution $F(s)$; it is not necessary that such vectors shall, as in the examples already given, have potential functions. Those that we have already given satisfy also certain restrictions of integrability which we denote by condition N, as follows:

Condition N. Any component of φ is summable superficially, and the normal component is summable along any curve of Γ; in particular, the integral of the absolute value of φ_n, along any curve of Γ or finite number of mutually exclusive curves remains finite, less than some fixed number N, provided that the total length of the curves remains finite, less than some fixed number s_0.

We have then the following theorems:

THEOREM III.—Let $G(s)$ be an arbitrary function of curves, additive and of limited variation,[7] and let $\psi(M)$ be a polarization vector for it which satisfies condition N; further let $u(M)$ be a function continuous over Σ' with its first partial derivatives. Then for every s of class Γ the following equation is valid:

$$\int_\sigma u dG(s) = \int_s u \psi_n ds + \int_\sigma \left(\frac{\partial u}{\partial x} \psi_x + \frac{\partial u}{\partial y} \psi_y \right) d\sigma \qquad (9)$$

THEOREM IV.—If $\psi(M)$ satisfies condition N and $u(M)$ is continuous and a potential function for its vector gradient in Σ, the equation

$$\int_\sigma udG(s) = \int_s u\psi_n ds + \int_\sigma (\nabla_x u\psi_x + \nabla_y u\psi_y)d\sigma \qquad (9')$$

remains valid provided that one of the following hypotheses (α), (β) or (γ) be imposed:

(α) $\nabla u(M)$ is bounded.

(β) $\psi(M)$ is bounded.

(γ) The quantities $\{\nabla u(M)\}^2$ and $\{\psi(M)\}^2$ are summable superficially.

If in this theorem we put $u(M) = \log 1/r$, we obtain:

THEOREM V.—If $\psi(M)$ satisfies condition N the equation

$$\int_\sigma \log \frac{1}{r}\, dG(s) = \int_s \log \frac{1}{r}\, \psi_n ds + \int_\sigma \left\{\psi_x \frac{\partial}{\partial x} \log \frac{1}{r} + \psi_y \frac{\partial}{\partial y} \log \frac{1}{r}\right\} d\sigma \qquad (10)$$

is valid, given the region σ, for all points M_1 except possibly those which form a point set of superficial measure zero.

One or two more special theorems are perhaps interesting. If in Theorem II we put $V = \log 1/r$, we obtain the equation

$$2\pi U(M_1) = \int_s \left\{U \frac{\partial}{\partial n} \log \frac{1}{r} - \log \frac{1}{r}\, \nabla_n u\right\} ds + \int_\sigma \log \frac{1}{r}\, dF(s) \qquad (11)$$

which holds whenever $U(M_1)$ is the derivative of its own superficial integral, and, therefore, except for the points of a set of superficial measure zero.

If in Theorem IV we put

$$\psi = \lambda\varphi$$

where λ is a scalar point function, so that the vector φ satisfies the equation

$$\int_s \lambda\varphi_n ds = G(s)$$

then equation (9′) reduces to the following:

$$\int_\sigma \lambda\{\nabla_x u\varphi_x + \nabla_y u\varphi_y\} d\sigma = -\int_s \lambda u\varphi_n d\sigma + \int_\sigma udG(s). \qquad (12)$$

These theorems provide for the Stieltjes integral with respect to a function of curves a representation in terms of a Lebesgue integral and a curvilinear integral. The curvilinear integral is essentially a curvilinear integral, depending only on the contour, and differs thus from the integral around general boundaries defined by P. J. Daniell,[9] which is a "frontier integral" not uniquely determined by the contour itself, but by some superficial point set.

The theorems of this section remain true if for σ is substituted a set of points measurable superficially in the sense of Borel, for s its frontier, and for $F(s)$, $G(s)$ the corresponding additive functions of point sets $f(e)$, $g(e)$, the frontier integrals being defined according to the method of Daniell. We shall come later to another possible method of determining frontier integrals.

3. *Definitions and Examples.*—It is desirable to render more precise some of the notions which we have used.

A curve of class Γ is a simple closed rectifiable curve composed of a fiinite number of pieces, each piece having a definite direction at every point; for each curve there is a constant Γ such that the following inequality holds.

$$\int_s \frac{|\cos nr|}{r_{M_1M}} ds < \Gamma,$$

wherever the point M_1 may be. For a curve which is everywhere convex, for instance, Γ may be taken as 2π.

With respect to gradient vector and potential function we have the fol lowing definition. If $A(M)$ is a vector point function whose components in two fixed directions (and therefore in any fixed direction) are summable, and $u(M)$ is a scalar point function whose integrals may be defined along any curve of class Γ, then A is spoken of as a *gradient* of u, and u as a *potential function* of A provided the equation

$$\int_\sigma A_\alpha d\sigma = \int_s u d\alpha' \qquad (13)$$

is satisfied for every s of Γ and for every α, the subscript α being used to represent a fixed direction and α' that direction advanced by $\pi/_2$.

It is sufficient that (13) should hold for two distinct directions α in order to hold for every direction.

The curvilinear integral in (13) is an example of what may be called an *absolutely continuous* function of curves. For the purpose of this article we say that an additive function of curves is of *limited variation* if it is the difference of two additive functions of curves, each one positive or zero for every curve of class Γ.

Every curve s of Γ determines a set e of points interior to the region σ which it bounds. We shall consider correspondences between functions of curves $F(s)$ and functions of points sets $f(e)$ such that for every curve on which $F(s)$ is continuous the equation

$$f(e) = F(s) \qquad (14)$$

holds. Given $F(s)$ it may be shown that $f(e)$ is uniquely determined over all sets measurable in the sense of Borel, simply by means of (14).

On the other hand the equation (14) is not sufficient to determine $F(s)$, given $f(e)$. The discontinuities of $F(s)$ may still be distributed in various ways. The $F(s)$ which is defined by the formula

$$F(s_1) = \frac{1}{2\pi}\int_{s_1} ds_1 \int_\Sigma \frac{\cos nr}{r} df(e) \qquad (15)$$

has all of its discontinuities of the *first kind;* that is, if a portion, s', which has everywhere a tangent, forms part both of s_1 and s_2, otherwise mutually exterior, a possible discontinuity upon s' would be shared equally by

$F(s_1)$ and $F(s_2)$; and if a point P is a vertex for s_1, a portion of the possible discontinuity at P proportional to the angle at the vertex is taken up by $F(s_1)$.

In order to illustrate this correspondence, take the following example. Let Σ be the square $0<x<1$, $0<y<1$; and to the points M^i of rational coördinates in this Σ assign positive numbers $p(M^i)$ in such a way that when the points are arranged in countable order, the series of numbers will be convergent; and make the definition

$$f(e) = p(e) = \Sigma_i p(M^i), \tag{16}$$

where the sum is extended over all the M^i in the set e. Introduce now an auxiliary function $\mu(\theta,M)$, which for a given point M depends on an angle θ made with a fixed direction, and which is such that the quantity $\int_0^{2\pi} \mu(\theta,M)d\theta$ is everywhere unity. Consider the quantity

$$H(M,\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2} \mu(\theta, M)d\theta.$$

For the purpose of defining $F(s)$ every point of s may be considered temporarily as a vertex, the tangents forward and backward from M making, respectively, angles θ_1,θ_2 with the fixed direction. Then the equation

$$F(s) = \Sigma'_\sigma p(M^i) + \Sigma_s H(M^i,\theta_1,\theta_2)p(M^i),$$

where the first summation takes all the points M^i interior to σ and the second takes all the points M^i on the boundary s, satisfies the equation of correspondence (14) and defines an additive function of curves of limited variation which is discontinuous on every curve which contains a rational point. In general, the discontinuities of $F(s)$ will not be of the first kind; in order to make them so, and thus satisfy the equation (15), it is sufficient to put $\mu(\theta,M) = 1/2\pi$.

The directional derivative of a point function $u(M)$ is not necessarily a vector, merely on the basis of its existence. If, however, this derivative for two directions, say $\partial u/\partial x$ and $\partial u/\partial y$, is summable superficially and satisfies for these two directions and all curves of Γ the equation (13), then the vector which has these two quantities for its components in the two directions will be a gradient vector, and the original function its potential function. Yet it is simpler for the purpose of proving theorems and less artificial, to use directly a generalized partial derivative which has itself vectorial properties.

We say that $D_\alpha u$, *the generalized derivative in the direction* α of $u(M)$, is the limit, if such limit exists, of the expression

$$D_\alpha u = \lim_{\sigma=0} \frac{1}{\sigma} \int_s u d\alpha' \tag{17}$$

where the fixed direction α' makes an angle of $\pi/2$ with the fixed direction α, and σ denotes the area enclosed by s; it is understood that σ tends

toward 0 only in such a way that the ratio σ/d^2, where d is the diameter of σ, remains different from zero by some positive quantity.

It is a theorem that the expression (6) is the generalized derivative of the expression (5) except at the points of a set E_o of superficial measure zero, which may be chosen independently of the direction α. The function defined by (5) is therefore a solution of the equation

$$\int_s D_n u \, ds = F(s) \tag{18}$$

for any curve which does not contain points of E_o of more than linear measure zero. The transition from this equation to the equation

$$\int_s \frac{\partial u}{\partial n} ds = F(s)$$

involving derivatives in the customary sense, is possible for classes of curves defined in terms of curvilinear coördinates, by means of the identity between a double and an iterated integral in these coördinates.

4. *A Double Integration by Parts.*—The Green's theorem from the point of view of section 2 may be regarded as an integration by parts of the multiple Stieltjes integral with respect to one dimension only. It is perhaps worth while to state the theorem which one obtains by a complete integration by parts of the double integral.

Let $u(M)$ be a continuous point function of limited variation[10] and let $q(M) = G(\bar{s})$ where $\bar{s}$ is the rectangle whose diagonally opposite vertices are M and the origin. Then if s is any simple closed regular curve, and points P_i with coördinates x_i, y_i are taken round the curve such that $|x_{i+1}-x_i|<\delta, |y_{i+1}-y_i|<\delta$ the following integrals have meaning:

$$\begin{aligned} \int u d_y q &= \lim_{\delta = o} \Sigma_i u(x_i,\eta_i)\left\{q(x_i,y_{i+1})-q(x_i,y_i)\right\} \\ &\qquad\qquad y_i \lesseqgtr \eta_i \lesseqgtr y_{i+1}, \\ \int u d_x q &= \lim_{\delta = o} \Sigma_i u(\xi_i,y_i)\left\{q(x_{i+1},y_i)-q(x_i,y_i)\right\} \\ &\qquad\qquad x_i \lesseqgtr \xi_i \lesseqgtr x_{i+1}, \end{aligned} \tag{19}$$

and the identity

$$\int_\sigma udq = \int_\sigma qdu + \int_\sigma dqu - \int_s ud_yq + \int_s ud_xq \tag{19'}$$

is valid.[11] In this equation the quantity $\int dqu$ stands for the total variation of qu over the region σ.

5. *Applications.*—Consider the function

$$W(z_1) = \frac{-1}{2\pi i}\int_\Sigma \frac{df(e)}{z-z_1} \tag{20}$$

where z denotes the quantity $x + iy$, where $f(e) = f_1(e) + if_2(e)$ is a complex additive function of point sets, and the integral has the meaning:

$$W(z_1) = U(M_1) + iV(M_1)$$

with

$$U(M_1) = \frac{1}{2\pi}\int \frac{y-y_1}{r^2}\, df_1(e) - \frac{x-x_1}{r^2}\, df_2(e)$$

$$V(M_1) = \frac{1}{2\pi}\int \frac{x-x_1}{r^2}\, df_1(e) + \frac{y-y_1}{r^2}\, df_2(e).$$

From this evaluation follows the equation

$$\int_{s_1} W(z_1)dz_1 = F(s_1), \tag{21}$$

where $F(s)$ is the function of curves with discontinuities of the first kind which is equivalent to $f(e)$.

If we extend the notion of generalized derivatives to functions of a complex variable, it turns out that what corresponds to the quantity

$$\frac{\partial W}{\partial x} + i\frac{\partial W}{\partial y}$$

is the limit:

$$\frac{\lim}{\sigma=0}\ \frac{1}{i\sigma}\int W(z)dz,$$

which is merely the superficial derivative, where it exists, of the additive function of point sets $f(e)/i$. But this quantity exists in Σ except possibly at the points of a set of superficial measure zero. At a point where this superficial derivative itself vanishes it seems opportune to follow the direction of Borel's concept, and speak of the function $W(z)$ as *monogenic*.

If $\kappa(z)$ is a function of a complex variable, analytic in Σ, and $Q(z)$ is a solution of the equation

$$\int_s Q(z)dz = F(s)$$

and equal to $W(z)$ plus an analytic function, the equation

$$\int_s \kappa Q dz = \int_\sigma \kappa dF(s)$$

is satisfied. In particular, by taking as $\kappa(z)$ the function $1/(z-z_1)$, it may be deduced that except possibly at the points of a set of superficial measure zero the following identity holds:

$$Q(Z_1) = \frac{1}{2\pi i}\int_s \frac{Q(z)}{z-z_1}\, dz + \frac{1}{2\pi i}\int_\sigma \frac{dF(s)}{z-z_1} \tag{22}$$

The function given by (20) is the unique derivative of a certain functional of open curves in the plane, considered as a function of the end-point of the curve. We have, in fact, $W(z) = dZ(1|z)/dz$, where

$$Z(1|z_1) = \frac{1}{2\pi i}\int_\Sigma \log\,(z-z_1)df(e). \tag{23}$$

Here the values of arctan $y-y_1/x-x_1$ are taken, say, as the principal values for a particular value of $z_1 = z_0$, thus defining log $(z-z_0)$ for z in Σ; log $(z-z_1)$ is then defined in general by means of the curve 1 which connects z_0 with z_1.

A different sort of application of these ideas is to the Dirichlet problem: to determine a harmonic function throughout a Weierstrassian region by assigned frontier values. For this purpose consider an arbitrary simply connected T region as defined by Osgood[12] whose boundary consists of more than one point; for simplicity we may as well restrict it to the finite domain.

Let M_o be an interior point of T and $g(M_o,M)$ its Green's function. Let $h(M_o,M)$ be the conjugate function of $g(M_o,M)$. These functions are both harmonic at every point of T, the point M_o excepted; and the following facts can be stated with reference to the frontier of T:

To an accessible point on the frontier corresponds a value of h, $h = C$ (and perhaps more than one value), but to two different accessible frontier points cannot correspond the same value C.[13] We see then that if a value of h corresponds to an accessible frontier point, the point is uniquely determined. But it can be shown that all values of h except possibly some which constitute a set of measure zero actually do correspond to accessible points on the frontier.

For a given point M_o of T we thus set up a correspondence between the curves $h\ (M_o,M) = C$ and the accessible points on the frontier of T.

Consider functions $u(M)$ which are limited and continuous in T, and whose first partial derivatives with their squares are summable over T. Form the integral

$$-\int_{P_1}^{P_2} u(M)\,\frac{\partial g}{\partial n}\,ds = \int_{h_1}^{h_2} u(M)dh(M_o,M) \tag{24}$$

along a curve joining a point, P_1, on $h = C_1$ to a point, P_2, on $h = C_2$ and composed of points M given by the relation $g(M_o,M) = \psi(h)$, where $\psi(h)$ is an arbitrary continuous function of values of h between C_1 and C_2. As $\psi(h)$ is let approach zero remaining always not-negative and less than some finite N, it is seen by comparison with additive functions of point sets that the integral (24) has a limiting value; and the Stieltjes integral in the right hand member defines in the limit an absolutely continuous function $U(h)$ of h_2, which is therefore the integral of its derivative with respect to h, this derivative being a summable function $\bar{u}(h)$. It may be shown that as M travels along a curve, $h = C$, we have $\lim u(M) = \bar{u}(C)$ except for values of C which constitute a set of linear measure zero. In other words, the frontier values of $u(M)$, taken in this sense, are summable with respect to h, and the frontier integral is properly a Stieltjes integral of frontier values.

Vice versa, given $\bar{u}$ limited and summable with respect to h on the frontier of T there is one and only one bounded function $u(M)$, continuous and harmonic in T, with first partial derivatives and their squares summable over T, and such that it takes on the frontier values $\bar{u}(h)$ "almost everywhere" in the above-described manner.

No single point M_0 need be preferred. If $\bar{u}(M)$ is summable with respect to $h(M_0,M)$ on the frontier of T when M_0 is some definite point in T, it is summable on the frontier of T with respect to $h(M_1,M)$, M_1 being any point of T.[14]

[1] *Proc. Amer. Acad. Sci., Boston,* **41,** 1905–06.

[2] Borel, E., *Leçons sur les fonctions monogènes,* Paris, 1917, p. 10.

[3] Evans, *Rend. R. Accad. Lincei,* (Ser. 5), **28,** 1919 (262–265).

[4] For example it is sufficient to consider merely rectangles. See also the class Γ below.

[5] de la Vallée Poussin, C., *Integrales de Lebesgue,* Paris, 1916, p. 57.

[6] Volterra, V., *Leçons sur les fonctions de lignes,* Paris, 1913, chapt. II. It is to be noted that the definitions of continuity in the case of functions of curves and in the case of functions of point sets do not correspond at all to the same situation.

[7] The discontinuities need not be of the first kind.

[8] By $\{\psi(M)\}^2$ is meant the quantity $\{\psi_x(M)\}^2 + \{\psi_y(M)\}^2$.

[9] *Bull. Amer. Meth. Soc.,* **25,** November, 1918 (65–68).

[10] de la Vallée Poussin, C., *loc. cit.,* p. 98; *Trans. Amer. Math. Soc.,* **16,** 1915 (493).

[11] This identity is given for the special case of a rectangle, where the integrals (19) do not have to be used, by W. H. Young, *Proc. London Math. Soc.,* (Ser. 2), **16,** 1917.

[12] Osgood, *Lehrbuch der Funktionentheorie,* Leipzig, 1912, p. 151.

[13] Osgood and Taylor, *Transactions of the American Mathematical Society,* **14,** 1913 (277–298).

[14] A complete exposition of the theorems of this paper will appear in an early issue of the *Rice Institute Pamphlet.*

THERMO-ELECTRIC ACTION AND THERMAL CONDUCTION IN METALS: A SUMMARY

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I shall now undertake to show in brief what I have accomplished in the series of papers which for some years I have been publishing in these PROCEEDINGS in relation to thermo-electric action and thermal conduction in metals.

Starting with the hypothesis of dual electric conduction, that is, conduction maintained in part by the passage of electrons from atomic union to atomic union during the contacts of atoms with positive ions, and in part by "free" electrons in the comparatively weak fields of force called the inter-atomic spaces, I derived[1] an equation giving the conditions of steady state in a detached bar hot at one end and cold at the other, and two equations for the conditions of equilibrium at a junction of two metals.

For the detached bar the steady state is a current of associated electrons up the temperature gradient and an equal current of free electrons down the temperature gradient. This involves a freeing of electrons from atomic unions, ionization, at the hot end of the bar, with absorption of heat, and re-association at the cold end, with evolution of heat. Evidently this is a process of conveying heat, and the question at once arises